GROUPOIDS

A function f is called a *binary operation* (or a composition mapping) on a set S if $f\colon S \times S \to S$.

A *groupoid* is an ordered pair $(S, \cdot)$ where $\cdot$ is a binary operation on S. The set S is called the *underlying set* of the groupoid and the function $\cdot$ is called the *groupoid operation*. We say that S is a groupoid under f. A groupoid is also called a binary algebra. The *order* of a groupoid $(S, \cdot)$ is $\text{card}(S)$.

If S is a finite set, $\{a_1, a_2, \ldots, a_n\}$, the *Cayley table* (or composition table) for the groupoid $(S, \cdot)$ is an $n \times n$ rectangular table whose (i, j) element is $a_i a_j$.

Let $(S, \cdot)$ be a groupoid.	
The operator $\cdot$ is	if
commutative	$ab = ba$ for all $a, b \in S$
associative	$(ab)c = a(bc)$ for all $a, b, c \in S$

Let A and B be groupoids and let h be a function such that $h(xy) = h(x)h(y)$ for all $x, y \in A$.	
If	then h is called
$h\colon A \longrightarrow B$	a homomorphism
$h\colon A \xrightarrow{\text{onto}} B$	an epimorphism
$h\colon A \xrightarrow{1\text{-}1} B$	a monomorphism
$h\colon A \xrightarrow[\text{onto}]{1\text{-}1} B$	an isomorphism
$h\colon A \longrightarrow A$	an endomorphism
$h\colon A \xrightarrow[\text{onto}]{1\text{-}1} A$	an automorphism

Groupoids A and B are *isomorphic* if there is an isomorphism $h\colon A \to B$.

Let $(S, \cdot)$ be a groupoid and $A \subseteq S$.	
Then A is	if
closed under $\cdot$	$AA \subseteq A$
a left ideal	$SA \subseteq A$
a right ideal	$AS \subseteq A$
an ideal	$SA \subseteq A$ and $AS \subseteq A$

An ideal is sometimes called a two-sided ideal.

Let $x \in S$ and $A \subseteq S$.	
Then	is called a
Sx	a principal left ideal
xS	a principal right ideal

TYPES OF GROUPOIDS

Let $(S, \cdot)$ be a groupoid.	
The groupoid is	if
commutative	$\cdot$ is commutative
Abelian	$\cdot$ is commutative
a semigroup	$\cdot$ is associative

Degenerate groupoids	
A groupoid S is called	if
null	$\text{card}(S) = 0$
trivial	$\text{card}(S) = 1$

SPECIAL GROUPOID ELEMENTS

Let $(S, \cdot)$ be a groupoid and $a \in S$.	
The element a is called	if for all $x, y \in S$
a left identity	$ax = x$
a right identity	$xa = x$
an identity	$ax = x$ and $xa = x$
left zero	$ax = a$
right zero	$xa = a$
zero	$ax = a$ and $xa = a$
left cancellable	$ax = ay \Rightarrow x = y$
right cancellable	$xa = ya \Rightarrow x = y$
cancellable	$ax = ay \Rightarrow x = y$ and $xa = ya \Rightarrow x = y$
an idempotent	$a^2 = a$

An identity is sometimes called a two-sided identity. A left zero is said to be left absorbent and a right zero is said to be right absorbent. A zero element is said to be absorbent, or two-sided absorbent.

Let $0 \in S$ be a zero element. Let $x \in S$ and $A \subseteq S$.	
Then	if
x annihilates A on the left	$xA = \{0\}$
x annihilates A on the right	$Ax = \{0\}$
x is nilpotent	$x^n = 0$ for some $n \in \mathbb{N}$
A is nilpotent	$A^n = \{0\}$ for some $n \in \mathbb{N}$

HALFGROUPOIDS

A *halfgroupoid* is an ordered pair (S, f) where f is a function such that $f\colon H \to S$ for some $H \subseteq S$. The *order* of S is $\text{card}(S)$. If $H \subseteq S$, then H is a *subhalfgroupoid* if H is a halfgroupoid such that $ab = c$ in S whenever $ab = c$ in H.

Let H be a subhalfgroupoid of G	
We say that	if
H is closed in G	$a, b \in H$, $ab = c$ in $G \Rightarrow$ $ab = c$ in H
G is an extension of H	(i) $ab = c$ in $G \Rightarrow a, b \in H$ (ii) $c \in G$, $c \notin H \Rightarrow c = ab$ in G for some $a, b \in H$
H generates G	the only closed subhalfgroupoid of G containing H as a subhalfgroupoid is H

Let G be an extension of H.	
We say that	if
G is complete	ab is defined in H for all $a, b \in G$
G is open	(i) $ab = c$ in G, $c \in H \Rightarrow ab = c$ in H (ii) $c \in G$, $c \notin H$, $ab = a'b' = c$ in $G \Rightarrow$ $a = a'$, $b = b'$

SUBGROUPOIDS

Let $(S, \cdot)$ be a groupoid and let $A \subseteq S$. If A is closed under $\cdot$, then A is called a subgroupoid of S.

If $f\colon A \to S$ and $g\colon B \to S$ are homomorphisms then we say g *extends* f to B if A is a subgroupoid of B and $f(a) = h(a)$ for all $a \in A$. We also say g induces f on A.

Let $\equiv$ be an equivalence relation on the set S. Then $\equiv$ is called a *congruence* for $\cdot$ if $a \equiv c$ and $b \equiv d \Rightarrow ab \equiv cd$. The equivalence classes are called *congruence classes*. The quotient sets are called *quotients*. The groupoid S/E under $\cdot/\equiv$ is called a *quotient groupoid* of S.

If $A \subseteq S$, we say that A is *normal* if $A = \varnothing$ or if there is a congruence E on S such that A is an E-class.

Let $\varnothing \neq A = A_0 \subset A_1 \subset \cdots \subset A_m = B$ be a finite chain of subgroupoids of S. Then this chain is called a *normal series* if A_{i-1} is normal in A_i for $1 \leq i \leq m$.

If $\mathcal{B}$ is a normal series from A to B and contains $\mathcal{A}$, then $\mathcal{B}$ is a *refinement* of $\mathcal{A}$. The chain $\mathcal{A}$ is *unrefinable* if it has no proper refinement.

The subgroupoid A is a *subnormal subgroupoid* of B if there is a normal series from A to B.

SEMIGROUPS

A semigroup is an associative groupoid.

Types of semigroups	
	a semigroup such that
A cancellation semigroup	every element is cancellable
A monoid	with an identity
A cyclic semigroup	every element is of the form c^n
A Gaussian semigroup	commutative cancellation semigroup S with identity such that every non-unit of S is a product of irreducible non-units in essentially one way
A GCD semigroup	a semigroup S such that any two elements $a, b \in S$ have a greatest common divisor
An LCM semigroup	a semigroup S such that any two elements $a, b \in S$ have a least common multiple
A PL-semigroup	a semigroup S such that $\mathcal{I}_S \cup \{\varnothing\}$ is closed under intersection and under union of chains.

If F is a semigroup and $\varnothing \neq B \subseteq F$, then F is called a *free semigroup* on B if each mapping f from B into a semigroup G can be extended to a unique homomorphism h of F into G. In this case, B is called a *basis* of F.

Divisibility in a semigroup S	
We say that	if for some $a \in S$
x divides y on the right	$y = ax$
x divides y on the left	$y = xa$

Divisibility in a commutative semigroup S	
We say that	if
$x\|y$	$y = ax$ for some $a \in S$
x divides y	$x\|y$
y is a multiple of x	$x\|y$
y is a left associate of x	$Sx = Sy$
y is a right associate of x	$xS = yS$
x is a unit	x is invertible
d is a common divisor of x and y	$d\|x$ and $d\|y$
m is a common multiple of x and y	$x\|m$ and $y\|m$
d is a greatest common divisor of x and y	$d\|x$ and $d\|y$ and $d^*\|x, d^*y \Rightarrow d^*\|d$
m is a least common multiple of x and y	$x\|m$ and $y\|m$ and $x\|m^*, y\|m^* \Rightarrow m\|m^*$
x is a divisor of y	$x\|y$
x if a factor of y	$x\|y$

Primes in a commutative semigroup S	
We say that x is	if for all $a, b \in S$
irreducible	$x = ab \Rightarrow x\|a$ or $x\|b$
prime	$x\|ab \Rightarrow x\|a$ or $x\|b$

MONOIDS

A monoid is a semigroup with an identity.

A *cyclic monoid* is a monoid that consists of the powers c^n of some element c. It is said to be *generated* by c.

QUASIGROUPS

If F is a monoid and $\varnothing \neq B \subseteq F$, then F is called a *free monoid* on B if each mapping f from B into a monoid G can be extended to a unique homomorphism h of F into G. B is called a *basis* of F.

A groupoid S is called a *quasigroup* if any two of $a, b, c \in S$ in $ab = c$ uniquely determines the third.

An *inversive quasigroup* is one in which $a, b \in S$ implies there is a unique $d \in S$ such that $da = b$ and for any $c \in S$, there is a $c' \in S$ such that for all $a \in S$, $(ac)c' = a$.

The *nucleus* of an inversive quasigroup is $\{x \cdot x' | x \in S\}$.

LOOPS

A *loop* is a quasigroup with an identity.

A *totally symmetric loop* is a commutative loop that satisifes $a(ab) = b$ for all $a, b \in S$.

GROUPS

A *group* is a monoid in which every element is invertible.

There are many equivalent definitions and many types of groups, but this survey is meant only to discuss structures simpler than a group.