SiAM 80-19. by E. Arthurs and B. W. Stuck

(a) Show that the following inequality holds:
\[ \sum_{i=1}^{N} G(S_i, A_i) \geq G \left( \sum_{i=1}^{N} S_i, \sum_{i=1}^{N} A_i \right), \]
where
\[ 0 \leq A_i < S_i, \quad S_i = 1, 2, \ldots, \quad i = 1, \ldots, N, \]
\[ G(S, A) = AB(S, A), \quad B(S, A) = \frac{A^2/2!}{\sum_{k=0}^{S} A^k/k!}. \]

(b) Show that the following inequality holds:
\[ \sum_{i=1}^{N} H(S_i, A_i) \geq H \left( \sum_{i=1}^{N} S_i, \sum_{i=1}^{N} A_i \right), \]
where
\[ 0 \leq A_i < S_i, \quad S_i = 1, 2, \ldots, \quad i = 1, \ldots, N, \]
\[ H(S, A) = \frac{A}{S - A} C(S, A), \]
\[ C(S, A) = \frac{A^2/(S-1)![(S-A)/A]}{\sum_{k=0}^{S-1} A^k/k! + A^2/(S-1)!(S-A)}. \]

AMM E2884. by Lawrence Harris

Let \( x_1, \ldots, x_n \) be distinct real numbers. Set
\[ S = \sum_{k=1}^{n} (1 + x_k^2)^{n/2} P(k), \]
where \( P(k) = \prod_{j \neq k} |x_k - x_j| \). Prove \( S \geq n \). When does \( S = n \)?

CRUX 992. by Harry D. Ruderman

Let \( \alpha = (a_1, a_2, \ldots, a_{mn}) \) be a sequence of positive real numbers such that \( a_i \leq a_j \) whenever \( i < j \), and let \( \beta = (b_1, b_2, \ldots, b_{mn}) \) be a permutation of \( \alpha \). Prove that
\[ \sum_{j=1}^{m} \prod_{i=1}^{n} a_{m(j-1) + i} \geq \sum_{j=1}^{m} \prod_{i=1}^{n} b_{m(j-1) + i}. \]

(b) \[ \prod_{j=1}^{m} \sum_{i=1}^{n} a_{m(j-1) + i} \leq \prod_{j=1}^{m} \sum_{i=1}^{n} b_{m(j-1) + i}. \]

BALKAN 1984/1.

Let \( x_1, x_2, \ldots, x_n, (n \geq 2) \) be positive numbers whose sum is 1. Prove that
\[ \frac{1}{1 + x_2 + x_3 + \cdots + x_n} + \frac{x_2}{1 + x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n}{1 + x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n}{2n - 1}. \]

GAZ 68.G. by H.-J. Seiffert

For integers \( m \geq 2 \), prove or disprove:
\[ \sum_{k=1}^{m} \frac{1}{2k + 1} \leq \frac{1}{\sqrt{m}} \leq \sum_{k=1}^{m} \frac{1}{2k + 1}. \]

MSJ 520. by F. David Hammer

Prove that if \( a_1, a_2, \ldots, a_n \) are real numbers such that \( a_1 + a_2 + \cdots + a_n = 1 \), then
\[ a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}. \]

POLAND 1984/3.

If \( a, x_1, x_2, \ldots, x_n \) (\( n \geq 2 \)) are positive real numbers, prove that
\[ \frac{x_1 - x_2}{x_1 + x_2} + \frac{x_2 - x_3}{x_2 + x_3} + \cdots + \frac{x_{n-1} - x_n}{x_n + x_1} \geq \frac{n^2}{2(x_1 + x_2 + \cdots + x_n)} \]
and determine when there is equality.

SPECT 15.9. by A. J. Douglas and G. T. Vickers

Let \( x_1, x_2, \ldots, x_n \) be real numbers such that \( 0 \leq x_i \leq 1 \) for \( i = 1, 2, \ldots, n \). Prove that
\[ \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \left\{ \begin{array}{ll} \frac{1}{4} - \frac{1}{4n^2} & \text{when } n \text{ is even} \\ \frac{1}{4} & \text{when } n \text{ is odd} \end{array} \right. \]
Discuss when equality occurs.

PENT 324. by Michael W. Ecker

Let \( x_1, x_2, \ldots, x_n \) be positive numbers whose sum is 1. What is the smallest possible value for the sum of the reciprocals
\[ \sum_{i=1}^{n} \frac{1}{x_i}? \]

SSM 3835. by Alan Wayne

Prove that if the sum of \( n \) positive numbers is 1, then the sum of their reciprocals is at least \( n^2 \).

NAW 571. by R. J. Strooker

For any real numbers \( x_1, \ldots, x_n, \) where \( n \geq 2 \), define
\[ F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ f(x_i - x_j) - f(x_i + x_j) \right\}. \]
Here \( f(x) \) is defined by
\[ f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases} \]
Show that \( F(x_1, x_2, \ldots, x_n) \geq 0 \). When does equality hold?

Inequalities: fractions

CANADA 1984/5.

Given any seven real numbers, prove that there are two of them, say \( x \) and \( y \), such that
\[ 0 \leq \frac{x - y}{1 + xy} \leq \frac{1}{\sqrt{3}}. \]