1994 Putnam problems and unofficial solutions, revised

As usual, first come the problems, then the problems with solutions. Comments and criticism at the end.

Send any followup remarks to the `USENET` newsgroup `sci.math`.

Problems

Problem A1

Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Problem A2

Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

Problem A3

Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

Problem A4

Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

Problem A5

Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}, \qquad \text{with } i_1 < i_2 < \cdots < i_{1994}.$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

Problem A6

Let $f_1, f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$F = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} \mid e_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in F map A to itself.

Problem B1

Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

Problem B2

For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

Problem B3

Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

Problem B4

For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\lim_{n\to\infty} d_n = \infty$.

Problem B5

For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

($\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the k-fold composition of f_α.)

Problem B6

For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100} \qquad \text{implies} \qquad \{a, b\} = \{c, d\}.$$

Unofficial solutions

Problem A1

Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Solution: If $\sum_{n\geq 1} a_n$ converges then

$$\sum_{n\geq 1} a_n = a_1 + \sum_{n\geq 1}(a_{2n} + a_{2n+1}) \geq a_1 + \sum_{n\geq 1} a_n > \sum_{n\geq 1} a_n,$$

which is absurd.

Problem A2

Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

Solution: Map the plane to itself by $f(x, y) = (3y, x/3)$. This map takes the y-axis to the x-axis, the ellipse $\frac{1}{9}x^2+y^2 = 1$ to itself, and the line $y = \frac{1}{2}x$ to the line $y = \frac{2}{9}x$. Hence it takes the region bounded by $y = \frac{1}{2}x$, the x-axis, and the ellipse to the region bounded by $y = \frac{2}{9}x$, the y-axis, and the ellipse. Finally, the Jacobian Df equals $\begin{pmatrix} 0 & 3 \\ 1/3 & 0 \end{pmatrix}$, which has determinant -1, so f preserves areas. Hence $m = 2/9$.

Problem A3

Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

Solution: Put the triangle into the real plane with vertices $A = (0, 1), B = (1, 0), C = (0, 0)$. Also write D, E, F, G for

$$(\sqrt{2} - 1, 2 - \sqrt{2}), (\sqrt{2} - 1, 0), (0, \sqrt{2} - 1), (2 - \sqrt{2}, \sqrt{2} - 1)$$

respectively. Finally write a, b, c, d, e, f, g for the colors of A, B, C, D, E, F, G respectively.

If $a = b$ then we are done: the distance $|A - B|$ is more than $2 - \sqrt{2}$. So assume $a \neq b$.

The points C, D, E, F, G are all at distance at least $2 - \sqrt{2}$ from A and from B. Hence we are done if any of c, d, e, f, g equals a or b. So assume that none of c, d, e, f, g equals a or b.

Now there are only two colors among c, d, e, f, g. So three of c, d, e, f, g are the same. Hence $c = d$ or $d = e$ or $e = f$ or $f = g$ or $g = c$. But the distances $|C-D|, |D-E|, |E-F|, |F-G|, |G-C|$ are all at least $2 - \sqrt{2}$. So we are done.

Problem A4

Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Solution: If $XY = I$ with X, Y integer matrices then $\det X \det Y = 1$ so $\det X = \pm 1$. Hence $\det(A + kB) = \pm 1$ for $k = 0, 1, 2, 3, 4$. So $\det(A + kB)$ repeats one value at least

three times. But

$$\det(A + kB) = \det A + k \operatorname{tr}(A^{-1}B) \det A + k^2 \det B.$$

This is a quadratic in k which takes the same value three times, so it is constant. Hence $\det(A + kB) = \det A = \pm 1$ for every k.

Problem A5

Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}, \qquad \text{with } i_1 < i_2 < \cdots < i_{1994}.$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

Solution: Write R for the set $\{r_n\}$. Since $\lim_{n \to \infty} r_n = 0$ there are, for any $h > 0$, only finitely many elements of R which are bigger than h.

Write $R_k = \sum_{1 \leq i \leq k} R$ for the set of sums of k elements, not necessarily distinct, of R. Then $S \subseteq R_{1994}$.

We prove by induction on $k \geq 0$ that any nonempty interval (a, b) contains a nonempty subinterval (c, d) which does not intersect R_k.

For $k = 0$ we have $R_k = \{0\}$, so we can take $(c, d) = (a, b)$ if $0 \notin (a, b)$ and $(c, d) = (0, b)$ if $0 \in (a, b)$.

Assuming that we have a nonempty interval (c, d) which does not intersect R_k, we now construct a nonempty subinterval (c', d') which does not intersect R_{k+1}.

Put $c' = (c + d)/2$, so that $c < c' < d$. Consider any $x \in R_{k+1}$ such that $c' < x < d$. Write $x = s_0 + s_1 + s_2 + \cdots + s_k$ with each $s_i \in R$. For any i we have $x - s_i \in R_k$, and $x - s_i < x < d$, so by the assumption $x - s_i \leq c$. Hence

$$s_i \geq x - c > c' - c.$$

There are only finitely many elements of R bigger than $c' - c$. So there are only finitely many choices of each s_i. Hence there are only finitely many possibilities for x. Let d' be the minimum such x, or d if no such x exists. Then, by construction, $x \geq d'$ for any $x \in R_{k+1}$ with $c' < x$. And that's what we wanted to prove.

Problem A6

Let $f_1, f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$F = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} \mid e_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in F map A to itself.

Solution: We prove the contrapositive. Let $f_1, f_2, \ldots, f_{10}$ be bijections of the integers. Let A be a nonempty finite set of integers. Let G be the set of $(e_1, \ldots, e_{10})$ such that $f_1^{e_1} \circ \cdots \circ f_{10}^{e_{10}}$ maps A to A. Say G has size at least 513.

We show first that f_1 maps A to A. There are 512 choices of $(e_2, e_3, \ldots, e_{10})$, and 513 elements of G, so at least two elements of G have the same $(e_2, e_3, \ldots, e_{10})$. There are only two possibilities for e_1, so both possibilities must appear. In other words both

$$f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}$$

and

$$f_1 \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}$$

map A to A. Write $h = f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}$. Now $h(A) = A$, and $f_1(h(A)) = A$, so $f_1(A) = A$ as desired.

By the same method we show for each i in turn that f_i maps A to A. There are 512 choices of $(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{10})$, so at least two elements of G have the same $(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{10})$. There are only two possibilities for e_i, so both possibilities must appear. In other words both

$$f_1^{e_1} \circ \cdots \circ f_{i-1}^{e_{i-1}} \circ f_{i+1}^{e_{i+1}} \circ \cdots \circ f_{10}^{e_{10}}$$

and

$$f_1^{e_1} \circ \cdots \circ f_{i-1}^{e_{i-1}} \circ f_i \circ f_{i+1}^{e_{i+1}} \circ \cdots \circ f_{10}^{e_{10}}$$

map A to A. Write $g = f_1^{e_1} \circ \cdots \circ f_{i-1}^{e_{i-1}}$ and $h = f_{i+1}^{e_{i+1}} \circ \cdots \circ f_{10}^{e_{10}}$. Then $g \circ h$ and $g \circ f_i \circ h$ both map A to A. By the inductive hypothesis, g maps A to A. Thus f_i maps $h(A) = g^{-1}(A) = A$ to A as desired.

So every f_i maps A to A. If $0 \in A$, select $n \notin A$; it is impossible for a composition of f_i's to take 0 to n. If $0 \notin A$, select $n \in A$; it is again impossible for a composition of f_i's to take 0 to n.

Problem B1

Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

Solution: Let $S(n)$ be the set of perfect squares s such that n is within 250 of s. For example

$$S(1) = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225\}.$$

Let $f(n)$ be the size of $S(n)$. For example $f(1) = 16$.

The difference between $S(n-1)$ and $S(n)$ is that we lose $n-251$ if $n-251$ is a square, i.e., at

$$n \in \{251, 252, 255, 260, 267, 276, 287, 300, 315, 332, 351, \ldots\},$$

and we gain $n+250$ if $n+250$ is a square., i.e., at

$$n \in \{6, 39, 74, 111, 150, 191, 234, 279, 326, 375, \ldots\}.$$

If neither $n-251$ nor $n+250$ is a square then $S(n) = S(n-1)$ and $f(n) = f(n-1)$.

We thus calculate successively $f(6) = 17$, $f(39) = 18$, $f(74) = 19$, $f(111) = 20$, $f(150) = 21$, $f(191) = 22$, $f(234) = 23$, $f(251) = 22$, $f(252) = 21$, $f(255) = 20$, $f(260) = 19$, $f(267) = 18$, $f(276) = 17$, $f(279) = 18$, $f(287) = 17$, $f(300) = 16$, $f(315) = 15$, $f(326) = 16$, $f(332) = 15$, and $f(351) = 14$. Hence the integers from 315 through 325 and the integers from 332 through 350 are the only integers up through 351 which are within 250 of exactly 15 perfect squares.

There are no such integers past 350. Indeed, if n is within 250 of $m^2, (m+1)^2, \ldots, (m+14)^2$ then $(m+14)^2 - m^2 \leq 500$ so $28m + 196 \leq 500$ so $m \leq 304/28 < 11$ so $m \leq 10$. Hence $n \leq m^2 + 250 \leq 100 + 250 = 350$.

Problem B2

For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

Solution: A straight line $ax + b$ intersects $x^4 + 9x^3 + cx^2 + 9x + 4$ in four distinct points if and only if $x^4 + 9x^3 + cx^2 + (9-a)x + (4-b)$ has four distinct roots. Write $g(x) = x^4 + 9x^3 + cx^2 + (9-a)x + (4-b)$.

If g has four distinct roots then $g' = 4x^3 + 27x^2 + 2cx + (9-a)$ has three distinct roots so $g'' = 12x^2 + 54x + 2c$ has two distinct roots. Hence the discriminant $(54)^2 - 4 \cdot 12 \cdot 2c$ is positive. In other words $c < (54)^2/96 = 243/8$.

Conversely, if $c < 243/8$ then g'' has two distinct roots, so g' has two distinct extreme points. We can choose a so that g' has a root between these two extreme points. Then g' has three distinct roots. Similarly we can choose b so that g has four distinct roots.

Problem B3

Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

Solution: For $k \geq 1$ the stated property does not hold. Consider the function $f(x) = \exp(x - 10 + \tan^{-1} x)$, where $\tan^{-1}$ means the angle between $-\pi/2$ and $\pi/2$. The derivative of f is

$$f'(x) = \exp(x - 10 + \tan^{-1}(x)) \left(1 + \frac{1}{1+x^2}\right) > \exp(x - 10 + \tan^{-1}(x)) = f(x).$$

But $f(x) < \exp(x - 10 + \pi/2) < e^x \leq e^{kx}$.

For $k < 1$ the stated property does hold. Say $f'(x) > f(x)$. Write $g(x) = f(x)e^{-x}$. Then $g'(x) = -f(x)e^{-x} + f'(x)e^{-x} > 0$. So $g(x) > g(0)$ for $x > 0$. In other words $f(x) > f(0)e^x$ for $x > 0$. If also $x > -(\log f(0))/(1-k)$ then

$$f(x) > f(0)e^x = f(0)e^{kx}e^{(1-k)x} > f(0)e^{kx}e^{-\log f(0)} = e^{kx}.$$

So let N be 0 or $-(\log f(0))/(1-k)$, whichever is larger.

Problem B4

For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\lim_{n\to\infty} d_n = \infty$.

Solution: The following solution is stolen from Robin J. Chapman.

Observe that matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ are closed under multiplication by $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. In particular $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^n$ must be of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$. Its determinant is $(-1)^n$, so $a^2 - 2b^2 = (-1)^n$. But $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^2$, so

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^n - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^2 + 2b^2 - 1 & 2ab \\ 4ab & a^2 + 2b^2 - 1 \end{pmatrix}.$$

If n is odd then $a^2 - 2b^2 = -1$ so $a^2 + 2b^2 - 1 = 2a^2$ so all entries are divisible by a. If n is even then $a^2 - 2b^2 = 1$ so $a^2 + 2b^2 - 1 = 4b^2$ so all entries are divisible by b. Both a and b increase as $n \to \infty$.

Problem B5

For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

($\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the k-fold composition of f_α.)

Solution: We will show that $\alpha = 1 - 1/n^2$ works.

Consider the function $g(z) = e^z(1-z)$ for $z \geq 0$. We have $g(0) = 1$ and $g'(z) = -ze^z \leq 0$, so $g(z) \leq 1$. Hence $1 - z \leq e^{-z}$. In particular $1 - 1/n^2 \leq e^{-1/n^2}$.

Now consider the function $h(z) = e^{-z}(1+z)$ for $z \geq 0$. We have $h'(z) = -ze^{-z} < 0$ for $z > 0$ so $h(z) < h(0) = 1$. Hence $e^{-z} < 1/(1+z)$. In particular $e^{-1/n} < 1/(1+1/n) = n/(n+1) = 1 - 1/(n+1)$. Hence

$$\left(1 - \frac{1}{n^2}\right)^n \leq e^{-1/n} < 1 - \frac{1}{n+1}.$$

So $\alpha^n(n^2-1) < n^2-n$.

Suppose that $\alpha^k(n^2-1) \geq n^2-k$ for some $k \leq n$. Then $\alpha^{k+1}n^2 \geq \alpha(n^2-k) \geq n^2-k-1$. So, iterating, we find that $\alpha^n(n^2-1) \geq n^2-n$; contradiction. Hence $\alpha^k(n^2-1) < n^2-k$ for each k. In other words $\alpha^{k+1}(n^2) < n^2-k$. Hence $f_{\alpha^{k+1}}(n^2) \leq n^2-k-1$.

We now prove, by induction on k, that $f_\alpha^k(n^2) = n^2-k = f_{\alpha^k}(n^2)$ for $0 \leq k \leq n+1$.

For $k=0$ we have $f_\alpha^0(n^2) = n^2-0 = f_{\alpha^0}(n^2)$.

Assume that $f_\alpha^k(n^2) = n^2-k = f_{\alpha^k}(n^2)$. We will show that $f_\alpha^{k+1}(n^2) = n^2-k-1 = f_{\alpha^{k+1}}(n^2)$, provided that $k \leq n$.

First, $\alpha = 1-1/n^2 \geq 1-1/(n^2-k)$, so $\alpha(n^2-k) \geq n^2-k-1$, so

$$f_\alpha^{k+1}(n^2) = f_\alpha(f_\alpha^k(n^2)) = f_\alpha(n^2-k) \geq n^2-k-1.$$

Second, $f_\alpha^k(n^2) \leq \alpha^k n^2$, so $\alpha f_\alpha^k(n^2) \leq \alpha^{k+1}n^2$, so

$$f_\alpha^{k+1}(n^2) \leq f_{\alpha^{k+1}}(n^2).$$

Finally, we proved before that

$$f_{\alpha^{k+1}}(n^2) \leq n^2-k-1.$$

Hence both $f_\alpha^{k+1}(n^2)$ and $f_{\alpha^{k+1}}(n^2)$ are squeezed into n^2-k-1.

Problem B6

For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a,b,c,d \leq 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100} \qquad \text{implies} \qquad \{a,b\} = \{c,d\}.$$

Solution: Note that 101 is prime. We check that 2 is a generator of the units modulo 101. We have $2^{10} = 1024 \equiv 14 \pmod{101}$, so $2^{20} \equiv 196 \equiv -6$, so $2^{40} \equiv 36$, so $2^{50} \equiv 36 \cdot 14 = 504 \equiv -1$. The order of 2 is a divisor of 100, but not a divisor of 20 or 50, so it must be 100.

We have $n_a + n_b \equiv n_c + n_d \pmod{100}$ so $a+b \equiv c+d \pmod{100}$. We also have $n_a + n_b \equiv n_c + n_d \pmod{101}$ so $2^a + 2^b \equiv 2^c + 2^d \pmod{101}$. This is a quadratic equation in disguise. Multiply by 2^a and substitute $2^{a+b} \equiv 2^{c+d} \pmod{101}$:

$$\begin{aligned} 2^{2a} + 2^{c+d} - 2^{a+c} - 2^{a+d} &\equiv 0 \pmod{101} \\ (2^a - 2^c)(2^a - 2^d) &\equiv 0 \pmod{101}. \end{aligned}$$

Hence $2^a \equiv 2^c$ or $2^a \equiv 2^d$. If $2^a \equiv 2^c \pmod{101}$ then $a \equiv c \pmod{100}$, since 2 is a generator. So $b \equiv d \pmod{100}$ and thus $a=c, b=d$. Similarly if $2^a \equiv 2^d \pmod{101}$ then $a=d, b=c$.

Comments

This is a revision of my 3 December 1994 solution set. The main improvement in this revision is Robin J. Chapman's vastly simpler solution to B4. I also fixed typos in A3, A4 (thanks to Par Kurlberg), and A5.

Note that there's quite a bit of slack in Problem B5. Chapman observes that k can be pushed up to $n\sqrt{2}$.

In past years each Putnam question envelope contained a summary sheet listing all the problems. Contestants would take the morning sheet, and their notes, to lunch, and discuss the morning questions. This year (and, Gerald Edgar points out, last year), the summary sheet was not handed out until the end of the exam. Contestants were not allowed to take any written material out of the exam room during lunch. What is the point of this absurd restriction?

As usual, several exam questions were poorly worded. The "set of 1024 functions" in A6 is a sequence, not a set. The triangle in A3 has "side length 1"; does this mean the (equal) legs, or does it mean the hypoteneuse? "Side lengths $1, 1, \sqrt{2}$" would have resolved the ambiguity. "At least a distance" in A3 should have been "a distance at least." Dave Savitt points out another question about A3: Does "triangle" mean the boundary or the interior?

Other complaints: In A6 "for each integer n" is missing a comma. Similarly for "if A is any nonempty finite set of integers" and, in B5, "for $1 \leq k \leq n$." These punctuation errors are distracting and potentially confusing. In B1 and B5 there are sentences starting with formulas. This is unconventional and, again, potentially confusing. Finally, in B5, how come the floor function "denotes" while the composition operator "is"? Should the reader's attention be focused on the notation in one case but the semantics in the other? I don't think Putnam contestants need this sort of distraction.

A3 is just like a problem on the 19th Putnam, and can be systematically solved in the same way. A5 seems a bit too standard for a Putnam problem. Overall, though, the exam wasn't too bad. A2 is a wonderful example of a problem that's much more difficult for people who know calculus than for people who don't. Similar praise goes to B4. (In my original solution set, I smashed through B4 using recurrence identities which took two hours to derive.) B6 is perhaps too easy, but beautiful nevertheless.

—Daniel J. Bernstein, `djb@silverton.berkeley.edu`, 7 December 1994